Marking Scheme

Combined maths

Grade 12

Part A

1) $\log_a b + \log_b a^2 = 3$

$\log_a b + 2\log_b a = 3$ (5)

Let $t = \log_a b$, $t + \frac{2}{t} = 3$

$t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0$

$t = 2$ or 1 (5)

when $t = 2$

$\log_a b = 2 \Rightarrow b = a^2$ (5)

when $t = 1$

$\log_a b = 1 \Rightarrow b = a$ this is impossible ($\because b \neq a$) (5)

2) $\frac{x^2+54}{x} > 15$

$\frac{x^2+54}{x} - 15 > 0$

$\frac{x^2-15x+54}{x} > 0$

$\frac{(x-9)(x-6)}{x} > 0$ (5)

$x-9$	-	-	-	+
$x-6$	-	-	+	+
x	-	+	+	+

$-\alpha$ X 0 ✓ 6 x 9 ✓ α(5)

$0 < x < 6$ and $9 < x < \alpha$

(5) (5)

(25)

3) $\lim_{x\to\alpha} \frac{1-\cos(x^2+bx+c)}{(x\to\alpha)^2}$

$= \lim_{x\to\alpha} \frac{2\sin^2 \frac{x^2+bx+c}{2}}{(x-c)^2}$ (10)

$= \lim_{x\to\alpha} \frac{\sin^2 \frac{(x-\alpha)(x-\beta)}{2}}{\left[\frac{(x-\alpha)(x-\beta)}{2}\right]^2} \times \frac{(x-\beta)^2}{2}$ (10)

$= (1)^{7} \times \frac{(1-\beta)^{7}}{8}$ (5)

$= \frac{(1-\beta)^{7}}{8}$

4) $f(x) = x^4 - 3x^2 + ax + b$

$f(1) = 0$ (5)

$\Rightarrow 1 - 3 + a + b = 0$

$a + b = 2$ .... (1) (5)

$f(3) = f(-1) + 72$ (5)

$81 - 27 + 3a + b = 1 - 3 - a + b + 72$

$4a = 16$

$a = 4$ (5)

(1) $\Rightarrow$ $b = -2$ (5)

5) $Y = \text{Cos}^2\theta + \text{Sin}^4\theta$

$Y = \text{Cos}^2\theta + \text{Sin}^2\theta\ (1 - \text{Cos}^2\theta)$ (5)

$= 1 - \text{Sin}^2\theta\,\text{Cos}^2\theta$ (5)

$= 1 - \frac{1}{4}\text{Sin}^2 2\theta$ (5)

$Y_{max} = 1$ (5) $Y_{min} = \frac{3}{4}$ (5)

6) $(\underline{a} + 2\underline{b}) . (5\underline{a} - 4\underline{b}) = 0$

$5|\underline{a}|^2 + 6\underline{a} . \underline{b} - 8|\underline{b}|^2 = 0$

$\underline{a} . \underline{b} = \frac{1}{2}$ (5)

$|\underline{a}||\underline{b}|\text{Cos}\,\theta = \frac{1}{2}$

$\text{Cos}\,\theta = \frac{1}{2}$

$\theta = \pi/3$

angle between $\underline{a}$, $\underline{b}$ is $\pi/3$ (5)

$|\underline{a} + 2\underline{b}|^2 = (\underline{a} + 2\underline{b}) . (\underline{a} + 2\underline{b})$

$= |\underline{a}|^2 + 4\underline{a}\underline{b} + 4|\underline{b}|^2$

$= 5 + 2 = 7$ (5)

$|\underline{a} + 2\underline{b}| = \sqrt{7}$ units

$|5\underline{a} - 4\underline{b}|^2 = (5\underline{a} - 4\underline{b}) . (5\underline{a} - 4\underline{b})$

$= 25|\underline{a}|^2 - 40\underline{a} \cdot \underline{b} + 16|\underline{b}|^2$

$= 41 - 20$

$= 21$

$|5\underline{a} - 4\underline{b}| = \sqrt{21}$ units (5)

(25)

7) $R^2 = P^2 + (2P)^2 - 2P(2P)\cos\theta$

$(\sqrt{3}P)^2 = 5P^2 - 4P^2 \cos\theta$ (5)

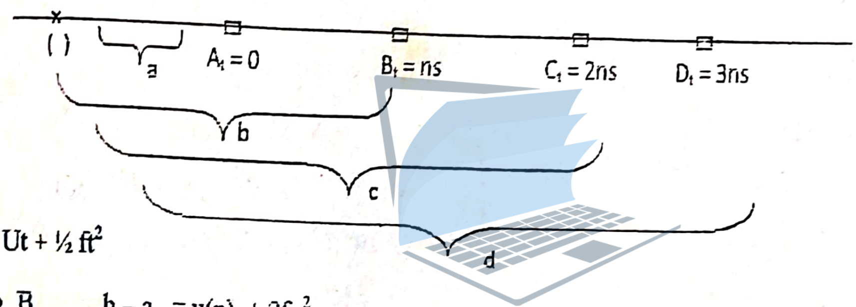
$\cos\theta = \frac{1}{2}$ (5)

$\theta = \pi/3$ Angle between forces is $\pi/3$ (5)

$\tan\alpha = \frac{2P\sin\pi/3}{P + 2P\cos\pi/3} = \frac{\sqrt{3}}{2}$ (5) $\Rightarrow \alpha = \tan^{-1}(\frac{\sqrt{3}}{2})$ (5)

(25)

8)



$S = Ut + \frac{1}{2}ft^2$

A → B $b - a = u(n) + 2fn^2$ .... (1) (5)

A → C $c - a = 2un + 2fn^2$ .... (2) (5)

A → D $d - a = 3un + \frac{9}{2}fn^2$ .... (3) (5)

(2) - (1) $c - b = nu + 3/2\, fn^2$ .... (4) (5)

(3) + (4) ⇒ $d - a = 3(c - b)$ (5)

(25)

$x = 2 + \sqrt{3}\cos 30 + 5\cos 60 - 2\cos 60$ (5)

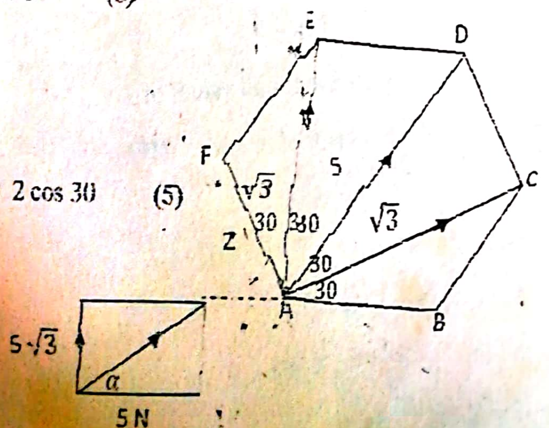
$= 2 + \frac{3}{2} + \frac{5}{2} - 1$

$= 5N$ (5)

$y = \sqrt{3} + 5\cos 30 + 3\sqrt{3}\cos 60 + 2\cos 30$ (5)

$= \sqrt{3} + \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sqrt{3}$

$= 5\sqrt{3}$ (5)



$\sqrt{x^2 + y^2} = \sqrt{75 + 25} = 10N$

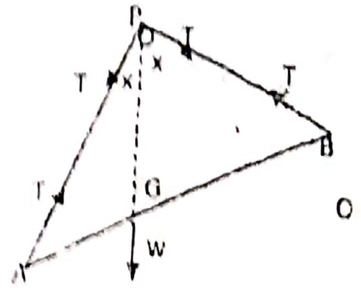
$\tan\alpha = \frac{5\sqrt{3}}{5} = \sqrt{3}$

$\alpha = 60°$

10) In equilibrium position

$\frac{AP}{PB} = \frac{AG}{GB} = \frac{1}{1}$ (5)

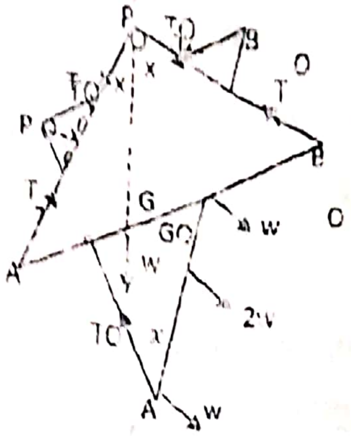
$AP = BP = 2l$ (5)



In equilibrium position

$\frac{AP}{BP} = \frac{AG_0}{BG_0} = \frac{1}{3}$ (5)

$AP = l$, $B_P = 3l$ (5)



length of sliding string $= 2l - l$

$= l$

(25)

Part B

11) a) $x^2 - 18x + 45$

$\alpha^2 - 1 + \beta^2 - 1 = 18$ (5)

$\alpha^2 + \beta^2 = 20$

$(\alpha + \beta)^2 - 2\alpha\beta = 20$

$(\alpha + \beta)^2 = 36$ (5)

$\alpha + \beta = \pm 6$ (5)

$\alpha + \beta = 6$ (5) $(\alpha > 0,\ \beta > 0)$

$(\alpha^2 - 1)(\beta^2 - 1) = 45$ (5)

$\alpha^2\beta^2 - (\alpha^2 + \beta^2) + 1 = 45$

$\alpha^2\beta^2 - 20 + 1 = 45$

$\alpha^2\beta^2 = 64$ (5)

$\alpha\beta = \pm 8$

$\alpha\beta = 8 \quad (\because \alpha > 0,\ \beta > 0)$ (5)

The equation whose root s are α an β is

$x^2 - x(\alpha + \beta) + \alpha\beta = 0$ (5)

$x^2 - 6x + 8 = 0$ (5)

(45)

b) $f(x) = -x^2 - 2(k - 1)x - 9$

When $f(x)$ is negative for all values of x

$\Rightarrow \Delta < 0$ and coefficient of $x < 0$ (10)

$\Rightarrow 4(k - 1)^2 - 36 < 0$ (5)

$\Rightarrow (k - 1)^2 - 9 < 0$

$\Rightarrow (k - 1 - 3)(k - 1 + 3) < 0$

$\Rightarrow (k - 4)\ (k + 2) < 0$ (5)

$\Rightarrow -2 < k < 4$ (5)

c) $P(x) \equiv (3x^2 + 5x - 2)\ f(x) + 3x + 5$ .... (1) (10)

$Q(x) \equiv (x^2 - 4)\ g(x) + x + 3$ .... (2) (10)

$P(x) + Q(x) \equiv (3x-1)(x+2) f(x) + (x^2 - 4) \times g(x) + 4(x+2)$ (10)

$\equiv (x+2)\,[(3x-1) f(x) + (x-2) g(x) + 4]$

So $(x+2)$ is a factor of $p(x) + Q(x)$ (10)

$P(x) - Q(x) \equiv (3x-1)(x+2) f(x) - (x^2 - 4) g(x) + 2x + 2$ (10)

When $P(x) - Q(x)$ is divided by $(x+2)$

the Remauder is $= P(-2) - Q(-2)$ (10)

$= 0 - 0 - 4 + 2 = -2$ (10)

(80)

(150)

12) a)

i) $\lim_{x\to\alpha} \{x - \sqrt{x^2 - 4030x}\}$

$= \lim_{x\to\alpha} \frac{x^2-(x^2-4030x)}{x+\sqrt{x^2-4030x}}$ (10)

$= \lim_{x\to\alpha} \frac{4030x}{x+\sqrt{x^2-4030x}}$

$= \lim_{x\to\alpha} \frac{4030}{1+\sqrt{1-\frac{4030}{x}}}$ (10)

$= \frac{4030}{2}$ (5)

$= 2015$

(25)

ii) $\lim_{x\to 0} \frac{2\cos x - 2 + x^2}{x^2}$

$= 2\lim_{x\to 0} \frac{(\cos x + 1)}{x^2} + 1$ (5)

$= 2\lim_{x\to 0} \frac{(1-2\sin^2\frac{x}{2} - 1)}{x^2} + 1$ (5)

$= -4\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \cdot \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \times ½ \times ½ + 1$ (10)

$= -1 + 1$ (5)

$= 0$

(25)

b) $\frac{4x-1}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-1)^2}$

$\Rightarrow A = -1, \quad B = 1, \quad C = 1$ (15)

$\frac{4x-1}{(x+2)(x-1)^2} = \frac{-1}{x+2} + \frac{1}{x-1} + \frac{1}{(x-1)^2}$ (5)

$$\frac{x^3+x+1}{(x+2)(x-1)^2} = \frac{(x^3-3x+2)+(4x-1)}{(x+2)(x-1)^2} \quad (10)$$

$$= \frac{(x+2)(x-1)^2+(4x-1)}{(x+2)(x-1)^2} \quad (10)$$

$$= 1 - \frac{1}{x+2} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \quad (10)$$

c) i) $\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq \sqrt{\frac{1}{a^2}\cdot\frac{1}{b^2}} = \frac{1}{ab}$ (10) (5)

$\frac{1}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{1}{bc}$ (5)

$\frac{1}{2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right) \geq \frac{1}{ca}$ (5)

Adding $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$ (5)

ii) $\frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}\right) \geq \sqrt{\frac{bc}{a}\cdot\frac{ca}{b}} = c$ (10) (25)

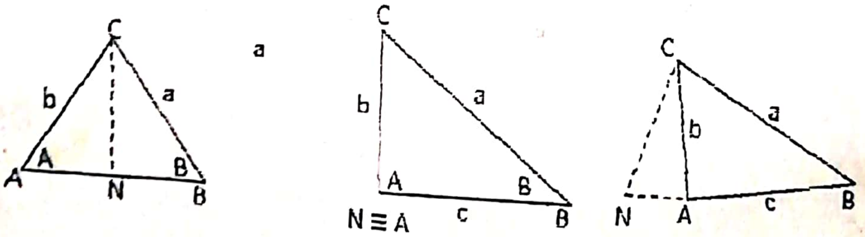
$\frac{1}{2}\left(\frac{ca}{b}+\frac{ab}{c}\right) \geq a$ (5)

$\frac{1}{2}\left(\frac{ab}{c}+\frac{bc}{a}\right) \geq b$ (5)

Adding $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c$ (5)

13) Sine rule in the usual notation : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (5) (25)

Proof: (5)



$CN = b \sin A$

$CN = a \sin B$ (5)

$CN = b$

$= b \sin \pi/2$

$= b \sin A$

$CN = a \sin B$ (5)

$CN = b \sin(\pi - A)$

$= b \sin A$

$CN = a \sin B$ (5)

$b \sin A = a \sin B$

$\frac{\sin A}{a} = \frac{\sin B}{b}$ (5)

Similarly $\frac{\sin A}{a} = \frac{\sin C}{c}$

$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (5)

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$

i) $\frac{a-b}{c} = \frac{K\sin A - k\sin B}{k\sin C}$ (5)

$= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2}}$ (10)

$= \frac{\sin\left(\frac{A}{2}-\frac{B}{2}\right)}{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}$ (10) $\left(\because \cos\left(\frac{A}{2}+\frac{B}{2}\right) = \cos\left(\pi/2 - \frac{C}{2}\right)\right.$

$= \sin\frac{C}{2}$

$\cos C/2 = \cos\left(\pi/2 - \frac{B}{2}\right)$

$\left.= \sin\frac{A+B}{2}\right)$

$= \frac{\sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}}$ (5)

$= \frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}}$ (5)

(35)

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ii) $\frac{a+b}{c} = \frac{K\sin A - k\sin B}{k\sin C}$ (5)

$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2}}$ (10)

$= \frac{\cos\left(\frac{A}{2}-\frac{B}{2}\right)}{\cos\left(\frac{A}{2}+\frac{B}{2}\right)}$ (10) $\left[\because \sin\left(\frac{A+B}{2}\right) = \sin\left(\pi/2 - \frac{C}{2}\right) = \cos\frac{C}{2}\right.$

$\left.\sin\frac{C}{2} = \sin\pi/2 - \left(\frac{A+B}{2}\right) = \cos\left(\frac{A+B}{2}\right)\right]$

$= \frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}}$ (5)

$= \frac{1+\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$ (5)

(35)

$\frac{a-b}{c} = \frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}}$

$(a-b)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = c\left(\tan\frac{A}{2} - \tan\frac{B}{2}\right)$ (5)

$(a-b-c)\tan\frac{A}{2} = (-c-a+b)\tan\frac{B}{2}$ (10)

$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \frac{c+a+b}{b+c-a}$ (5)

(20)

$\frac{a+b}{c} = \frac{1+\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$

$$(a+b)\left(1-\tan\frac{A}{2}\tan\frac{B}{2}\right)=c\left(1+\tan\frac{A}{2}\tan\frac{B}{2}\right)$$

$$a+b-c=(a+b+c)\tan\frac{A}{2}\tan\frac{B}{2} \quad (10)$$

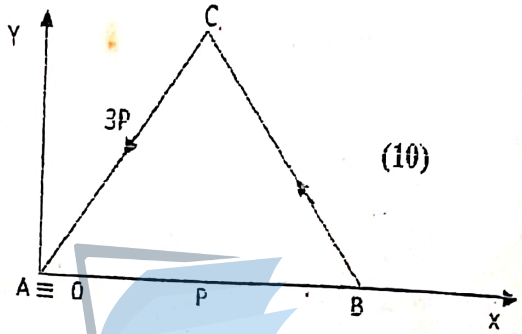
$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{a+b-c}{a+b+c} \quad (5)$$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}\cdot\tan\frac{A}{2}\tan\frac{B}{2}=\frac{c+a-b}{b+c-a}\cdot\frac{a+b-c}{a+b+c} \quad (5)$$

$$\tan^2\frac{A}{2}=\frac{(a+b-c)(c+a-b)}{(a+b+c)(b+c-a)} \quad (5)$$

$$\tan\frac{A}{2}-\sqrt{\frac{(a+b-c)(c+a-b)}{(a+b+c)(b+c-a)}}$$

15)



(10)

$\rightarrow X=P-5P\cos 60$ (15)

$=-\frac{3P}{2}$

$\leftarrow X=\frac{3P}{2}$ (10)

$\uparrow Y=2P\sin 60-3P\sin 60$ (15)

$=-P\sin 60$

$=-\sqrt{3}P/_2$

$\downarrow Y=+\sqrt{3}P/_2$ (10)

$\tan\alpha=\frac{\sqrt{3}P/_2}{3P/_2}$

$=\frac{1}{\sqrt{3}}$

$\alpha=\frac{\pi}{6}$ (10)

$R=\sqrt{\frac{9P^2}{4}+\frac{3P^2}{4}}=\sqrt{3}P$ (10)